Stationary states and correlations in (soft) active matter models

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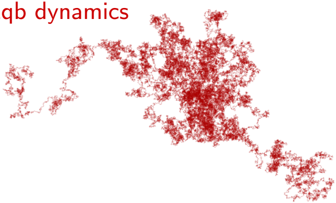
Fluctuating trajectories : passive *vs* active

- Stochastic process : system driven by a (temporally) randomly valued quantity.
- Passive matter : Eqb dynamics

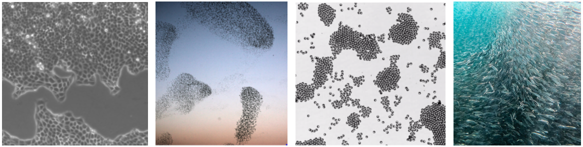
Brownian motion :



[Brown 1827]
[Einstein/Smoluchowski 1906]
[Langevin 1908]

→*e.g.* irregular motion of pollen grains (colloids) floating on water.

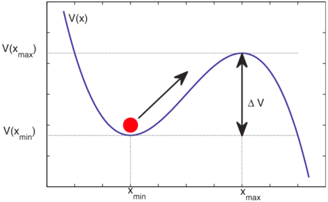
- Active matter : local dissipation of energy at agent's scales →Non eqb dynamics



→*e.g.* self propelled colloids in a bacterial bath (*i.e.* soft matter).

Stationary and escaping issues : motivations

→Equilibrium / Reversible dynamics in passive stochastic processes : basic toolbox



- [Boltzmann-Gibbs] Equilibrium distribution : $\mathbb{P}(x) \propto \mathrm{e}^{-V(x)/D}$
- [Arrhenius-Kramers] (low temperature regime) Reversible dynamics : $T_{\text{esc}} \propto \mathrm{e}^{\Delta V/D}$

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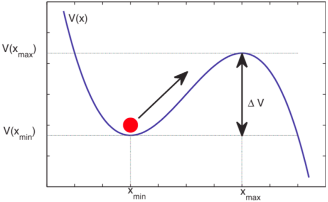

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Issues

- What are the **stationary properties** (*e.g.* escaping time, stationary distribution *etc.*) of active particles evolving in potential landscapes ?
- What **phenomena** do these properties underlie ?

Outline

1. Introduction : stochastic processes
2. Escaping active particle : uni-dimensional situation
    - Active particle toy model
    - Stationary states and escaping properties
3. Multi-dimensional situations / Weak activity
    - Situation
    - Trapping effects
    - Transverse ratchet effects
    - Rotational currents effects
4. Conclusion : contributions and openings

Modelisation : Active Ornstein-Uhlenbeck Particle (AOUP)

Langevin stochastic equation :

$$m\ddot{x} = -\gamma\dot{x} + \mathscr{F}_{\text{macro}} + \zeta(t)$$

$\zeta(t)$ random dissipation of energy

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- Correlations →Memory effect due to stochastic interactions with the bath

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Activity ↔ temporal correlations of the noise

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Overdamped limit (and $\gamma = 1$) →Active Ornstein-Uhlenbeck Particle :

$$\begin{cases} \dot{x} & = -V'(x) + v \\ \tau\dot{v} & = -v + \sqrt{2D}\,\xi(t) \end{cases}$$

$\xi(t)$: Gaussian white noise (Wiener process) $\langle\xi(t)\rangle = 0$ and $\langle\xi(t)\xi(t')\rangle = \delta(t-t')$

AOUP stationary distribution : pre-existing results

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potential barrier : Boltzmann-Gibbs equilibrium mapping.
ratchet effect : non-local, details of the whole V-landscape matter.

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force barrier, x_{cr} : position of the maximum force.

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Open questions

- **Stationary distribution** in between these two regimes (1d/2d) ?
- Characterisation of the possible undergoing **phase transition** ?

Perturbation theory / Systematic procedure

(1) Can we construct a local (*i.e.* Fokker-Planck based) method allowing us to **compute "all" τ^p-corrections** in the weak activity series expansion of Ψ ?

$$\Psi^{p_{\max}}(x,v,\tau) \underset{\tau\to 0}{\sim} \Delta V(x) + \sum_{p=1}^{p_{\max}} \Psi_p(x,v)\ \tau^p + \mathcal{O}\left(\tau^{p_{\max}+1}\right)$$

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Advantages / Drawbacks

1. Fully **under control** and systematic.
2. Access to all τ^p-corrections, given a $p_{\max}$.
3. Recover the two first corrections previously found by paths integrals and KSM methods.

→**But** not absolutely convergent expansion : zero-convergent radius ($R_\Psi = 0$)

τ-domain expansions / Resummation procedures

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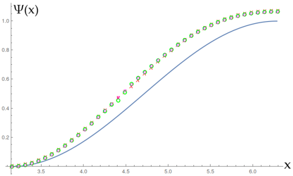

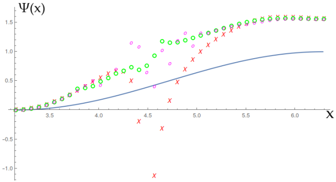

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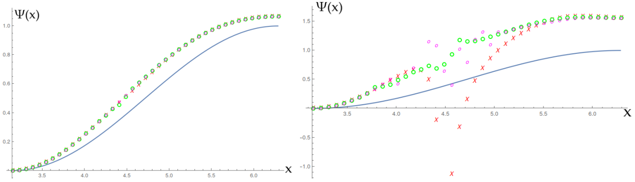

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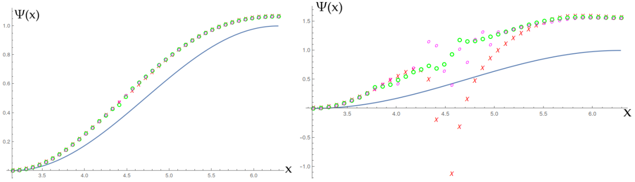

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Advantages / Drawbacks

- **Strong activity** regime "plateau" effect **captured**.
  →**But** Lack of validity in the x-space near the the maximal force location.

Situation

1d case

$$\Psi(x) = \Delta V(x) + \tau \frac{1}{2} V'(x)^2 - \tau^2 \frac{1}{2} \int_{x_{\text{min}}}^{x} \mathrm{d}x' \, V'(x')^2 V'''(x') + \tau^3 \frac{5}{12} V'(x)^3 V'''(x) - \tau^4 \ldots$$

- Add τ^p-corrections $\rightarrow R_\Psi = 0$.
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- Find τ^2**-correction** / **physical implications** due to $d = 2$.

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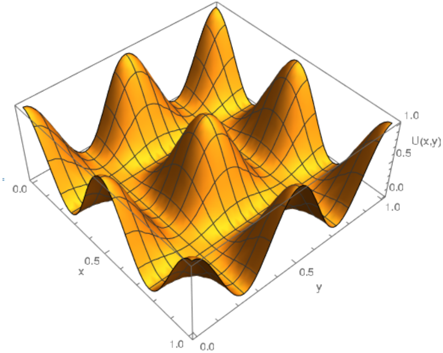
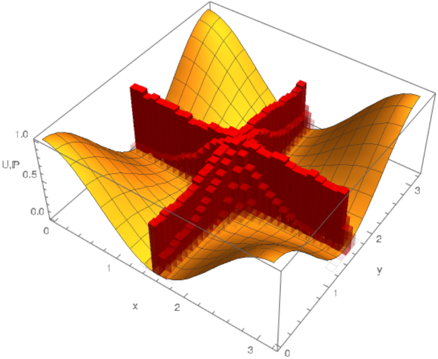
Highly non trivial generalisation of 1d procedure $\rightarrow$equation for Ψ_2 unsolvable $\forall V$.

2d equilibrium escape issue

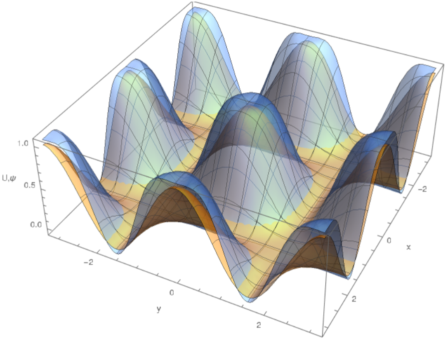
- $V(x,y) = \cos^2(x)\cos^2(y)$ (symmetric) →analytically solvable, up to τ^2 corrections.
- Equilibrium situation : white noise, $\tau = 0$ (no correlation).

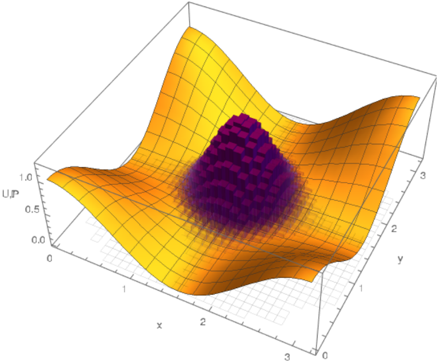




Localization by non-locality : 2d AOUP escape issue

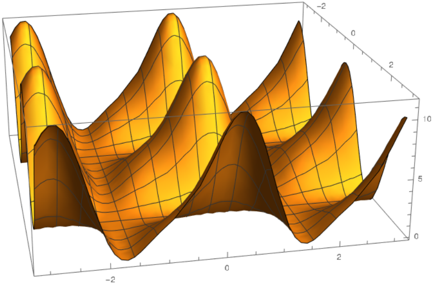
- Non-equilibrium situation : colored noise (self propulsion velocity) $\tau \neq 0$.
- Analytical predictions : non-local $\mathcal{O}(\tau^2)$ correction $\rightarrow \Psi$ inhomogeneous
  min of $\Psi \leftrightarrow$ max of $\mathbb{P}$ $\rightarrow$ Trapping effect.

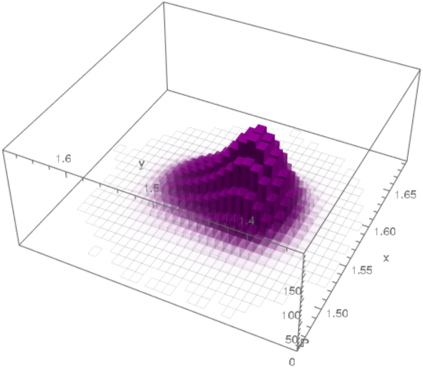




Ratchet by transverse non-locality : 2d AOUP escape issue

- Non-equilibrium situation : colored noise (self propulsion velocity) $\tau \neq 0$.
- $V(x,y)$ Asymmetric in the $\hat{\mathbf{y}}$ direction : local symmetry breaking.

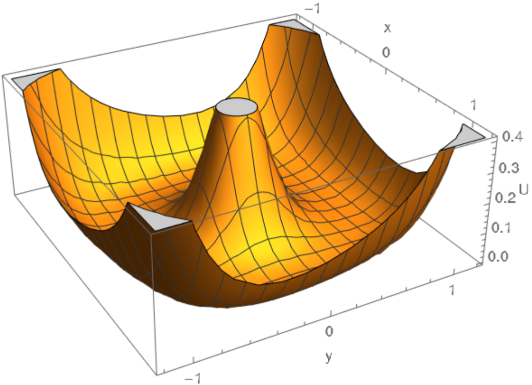




- →global flux : ratchet effect $\langle v_y \rangle \gg \langle v_x \rangle$: ∃ current.

Rotational current by non-locality : 2d AOUP escape issue

- Non-equilibrium situation : colored noise (self propulsion velocity) $\tau \neq 0$.
- $V(x,y)$: Broken rotational invariance



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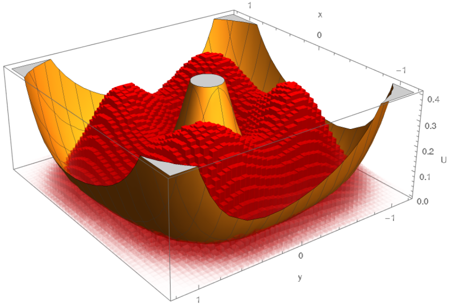

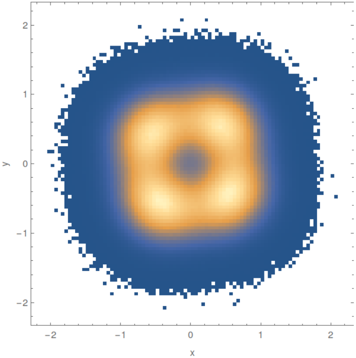

- Ratchet →rotational currents.

Activity →Equilibrium picture broken →non-local corrections to $\mathbb{P}$.

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New (fully controlled) procedures built

- Perturbative theory / **Systematic procedure** (weak activity regime) based on regularity conditions.
- Expansion of the τ-values range to reach the strong activity regime.

→Theoretical impacts :

- Theory of non-equilibrium metastability.
- Far from equilibrium classical statistical mechanics.

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New phenomena discovered

- 2d : **trapping effect** by non-locality, **ratchet effect** by transverse non-locality, **rotational currents** effect by non-locality.

→Potential implications :

- Sets of many metastable states, transit dynamics →Glasses theory.
- Disorder and trapping in granular media.
- Interface growth in active disordered media.

References

→Paths integrals approaches :

- « Nonlocal stationary probability distributions and escape rates for an active Ornstein-Uhlenbeck particle », E. Woillez, Y. Kafri, V.Lecomte, *J. Stat. Phys.*, [2020]
- « Bistability driven by correlated noise : Functional integral treatment », J. F. Luciani, A. D. Verga, *J. Stat. Phys.*, Vol. 50, **N° 3,4**, [1988]
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- « Path integrals and non-Markov processes II : Escape rates and stationary distributions in the weak-noise limit », J. McKane, A. J. Bray, T. J.Newman, *Phys. Rev. A*, Vol. 41, [1990]

→Fokker-Planck approaches :

- « Colored noise in dynamicalsystems », M. M. Kłosek-Dygas, B. J.Matkowsky, Z.Schuss, *J. Appl. Math*, Vol. 48, **N°2**, [1988]
- « Active Ornstein-Uhlenbeck particles », L. L.Bonilla, *Phys. Rev. E*, [2019]
- « Dynamical systems : A unified colored-noise approximation », P.Jung, P.Hänggi, *Phys. Rev. A*, Vol. 35, **N°10** [1987]

→Applications of AOUP :

- « Generalized Energy Equipartition in Harmonic Oscillators Driven by Active Baths », C. Maggi, M. Paoluzzi, N. Pellicciotta, A. Lepore, L. Angelani, R. Di Leonardo, *Phys. Rev. Lett.*, Vol. 113, [2014]
- « Equilibrium physics breakdown reveals the active nature of red blood cell flickering », H. Turlier *et al.*, *Nat. Phys.*, Vol. 12, [2016]

Supplementary material (1) $\mathscr{L}\mathscr{P}\mathscr{B}$-resummation

« Divergent series converge faster than convergent series because they don't have to converge »
F. Carrier

Quadratic equation :

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} + u^2(t) = 0$$

Laplace-Padé-Borel resummation : *[C. Allery et al.]*

$$\hat{u}(t) = \sum_{p=0}^{\infty} (-1)^p t^p$$

$\downarrow \mathscr{B}$

$$\sum_{p=0}^{\infty} \frac{(-1)^{p+1}}{p\,!} \xi^p \xrightarrow{\ \mathscr{P}\ } -\mathrm{e}^{-\xi}$$

$\uparrow \mathscr{L}$

$$1 + \int_0^{\infty} \mathrm{e}^{-\xi} \mathrm{e}^{-\xi/t}\, \mathrm{d}\xi = \frac{1}{1+t}$$

Supplementary material (2) 2d Analytical predictions

2d perturbation theory :

$$\Psi(\mathbf{r},\mathbf{v}) \underset{\tau \to 0}{\sim} \Delta U(\mathbf{r}) + \tau \Psi_1(\mathbf{r},\mathbf{v}) + \tau^2 \Psi_2(\mathbf{r},\mathbf{v}) + \mathcal{O}(\tau^3)$$

→Inspired by 1d non-singularity arguments (but highly non trivial generalisation) :

- $\mathcal{O}(\tau)$ correction

$$\Psi_1(\mathbf{r},\mathbf{v}) = \frac{\mathbf{v}^2}{2} + \frac{1}{2}(\nabla V)^2$$

- $\mathcal{O}\left(\tau^2\right)$ correction

$$\Psi_2(\mathbf{r},\mathbf{v}) = \frac{1}{2}\mathbf{v} \cdot \mathbb{H}\mathrm{ess}_V(\mathbf{r})\ \mathbf{v} + \mathcal{T}_2(\mathbf{r})$$

where $\mathcal{T}_2(\mathbf{r})$ such that

$$\nabla \mathcal{T}_2(\mathbf{r}) \cdot \nabla V(\mathbf{r}) = \mathcal{F}(\mathbf{r})$$

with $\mathcal{F}(\mathbf{r})$ known.